

## NEW EXTENSIONS FOR A THEOREM BY MOCANU

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ABSTRACT. For analytic functions  $f(z)$  in the open unit disk  $\mathbb{U}$  with  $f(0) = f'(0) - 1 = f''(0) = 0$ , P. T. Mocanu (Mathematica (Cluj), **42**(2000)) has considered some sufficient arguments of  $f'(z) + zf''(z)$  for  $|\arg(zf'(z)/f(z))| < \pi\mu/2$ . The object of the present paper is to discuss those problems for  $f(z)$  with  $f''(0) = f'''(0) = \dots = f^{(n)}(0) = 0$  and  $f^{(n+1)}(0) \neq 0$ .

## 1. INTRODUCTION

Let  $\mathcal{A}_n$  denote the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots \quad (n = 1, 2, 3, \dots)$$

that are analytic in the open unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$  and  $\mathcal{A} = \mathcal{A}_1$ .

Also, let  $\mathcal{H}[1, n]$  denote the class of functions  $f(z)$  of the form

$$f(z) = 1 + \sum_{k=n}^{\infty} a_k z^k \quad (n = 1, 2, 3, \dots)$$

which are analytic in  $\mathbb{U}$ .

Further, let the class  $\mathcal{STS}(\mu)$  of  $f(z) \in \mathcal{A}_n$  be defined by

$$\mathcal{STS}(\mu) = \left\{ f(z) \in \mathcal{A}_n : \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2}\mu, \ 0 < \mu \leq 1 \right\}$$

and  $\mathcal{S}^* = \mathcal{STS}(1)$ . This class  $\mathcal{STS}(\mu)$  was considered by Shiraishi and Owa [4].

Let  $f(z)$  and  $g(z)$  be analytic in  $\mathbb{U}$ . Then  $f(z)$  is said to be subordinate to  $g(z)$  if there exists an analytic function  $w(z)$  in  $\mathbb{U}$  satisfying  $w(0) = 0$ ,  $|w(z)| < 1$  ( $z \in \mathbb{U}$ ) and  $f(z) = g(w(z))$ . We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

## 2. LEMMAS

We need the following lemmas to consider our main results.

**Lemma 1.** *Let  $n$  be a positive integer,  $\lambda > 0$ , and let  $\beta_0 = \beta_0(\lambda, n)$  be the root of the equation*

$$\beta\pi = \frac{3}{2}\pi - \arctan(n\lambda\beta).$$

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In addition, let

$$\alpha = \alpha(\beta, \lambda, n) = \beta + \frac{2}{\pi} \arctan(n\lambda\beta)$$

for  $0 < \beta \leq \beta_0$ .

If  $p(z) \in \mathcal{H}[1, n]$  and

$$p(z) + \lambda zp'(z) \prec \left( \frac{1+z}{1-z} \right)^\alpha \quad (z \in \mathbb{U}),$$

then

$$p(z) \prec \left( \frac{1+z}{1-z} \right)^\beta \quad (z \in \mathbb{U}).$$

**Lemma 2.** Let  $q(z)$  be the convex function in  $\mathbb{U}$ , with  $q(0) = 1$  and  $\operatorname{Re}(q(z)) > 0$  for  $\mathbb{U}$ . Let the function  $h(z)$  be given by

$$h(z) = (q(z))^2 + nzq'(z) \quad (z \in \mathbb{U}).$$

If  $p(z) \in \mathcal{H}[1, n]$  and

$$(p(z))^2 + zp'(z) \prec h(z) \quad (z \in \mathbb{U}),$$

then  $p(z) \prec q(z)$  and this is sharp.

The above lemmas were given by Mocanu [3].

Applying Lemma 2, we obtain the following lemma.

**Lemma 3.** If  $p(z) \in \mathcal{H}[1, n]$  satisfies

$$|\arg((p(z))^2 + zp'(z))| < \phi(\mu) \quad (z \in \mathbb{U})$$

for

$$\phi(\mu) = \frac{\pi}{2}(\mu + 1) - \arctan \frac{\cos \frac{\mu\pi}{2}}{\sin \frac{\mu\pi}{2} + \frac{n\mu}{1-\mu} \left( \frac{1-\mu}{1+\mu} \right)^{\frac{1+\mu}{2}}}$$

and  $0 < \mu \leq 1$ , then

$$|\arg(p(z))| < \frac{\pi}{2}\mu \quad (z \in \mathbb{U}).$$

*Proof.* Let us define the function  $q(z)$  by

$$q(z) = \left( \frac{1+z}{1-z} \right)^\mu \quad (z \in \mathbb{U}) \tag{1}$$

for  $0 < \mu \leq 1$  and the function  $h(z)$  by

$$h(z) = (q(z))^2 + nzq'(z) \quad (z \in \mathbb{U}).$$

Then the function  $q(z)$  is convex in  $\mathbb{U}$  with  $\operatorname{Re}(q(z)) > 0$ ,  $h(z)$  is univalent in  $\mathbb{U}$  and  $h(\mathbb{U})$  is the symmetric domain with respect to the real axis.

If we set  $z = \exp(i\theta)$ ,  $0 \leq \theta < \pi$  and  $x = \cot \frac{\theta}{2}$ , then  $x \geq 0$ ,  $z = \frac{ix-1}{ix+1}$  and  $q(z) = (ix)^\mu$ . Hence

$$h(e^{i\theta}) = (ix)^{\mu-1} H(x),$$

where

$$H(x) = (ix)^{\mu+1} - \frac{n}{2}\mu(1+x^2).$$

Noting that  $\cos \frac{(\mu+1)\pi}{2} = -\sin \frac{\mu\pi}{2}$  and  $\sin \frac{(\mu+1)\pi}{2} = \cos \frac{\mu\pi}{2}$ , we see that

$$H(x) = P(x) + iQ(x),$$

where

$$\begin{cases} P(x) = -\sin \frac{\mu\pi}{2} x^{\mu+1} - \frac{n}{2}\mu(1+x^2) \\ Q(x) = \cos \frac{\mu\pi}{2} x^{\mu+1}. \end{cases}$$

Let

$$\varphi(\mu) = \min\{\arg(H(x)) : x \geq 0\}$$

and

$$\phi(\mu) = \varphi(\mu) + \frac{\pi}{2}(\mu - 1). \quad (2)$$

From (1) we deduce

$$\arg(h(e^{i\theta})) \geq \phi(\mu).$$

Since

$$G(x) = Q'(x)P(x) - P'(x)Q(x) = \frac{n}{2}\mu \cos \frac{\mu\pi}{2} x^\mu ((1-\mu)x^2 - (1+\mu)) = 0$$

has the root  $x_0 = \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}}$  and

$$\frac{Q(x_0)}{P(x_0)} = -\frac{\cos \frac{\mu\pi}{2}}{\sin \frac{\mu\pi}{2} + \frac{n\mu}{1-\mu} \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}},$$

we deduce

$$\varphi(\mu) = \pi - \arctan \frac{\cos \frac{\mu\pi}{2}}{\sin \frac{\mu\pi}{2} + \frac{n\mu}{1-\mu} \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}}. \quad (3)$$

Hence

$$|\arg(h(e^{i\theta}))| \geq \phi(\mu) \quad (-\pi < \theta < \pi)$$

where  $\phi(\mu)$  is given by (2) and (3).

From the assumption,

$$(p(z))^2 + zp'(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Hence by Lemma 2 we deduce  $p(z) \prec q(z)$ .

So, we obtain

$$|\arg(p(z))| < \frac{\pi}{2}\mu \quad (z \in \mathbb{U}).$$

□

## 3. MAIN RESULTS

Using Lemma 1 and Lemma 3, we get the following result.

**Theorem 1.** *If  $f(z) \in \mathcal{A}_n$  satisfies*

$$|\arg(f'(z) + zf''(z))| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U})$$

*for*

$$\alpha = \beta + \frac{2}{\pi} \arctan(n\beta),$$

$$\beta = \gamma + \frac{2}{\pi} \arctan(n\gamma),$$

$$\frac{\pi}{2}(\alpha + \gamma) \leq \phi(\mu)$$

*and*

$$\phi(\mu) = \frac{\pi}{2}(\mu + 1) - \arctan \frac{\cos \frac{\mu\pi}{2}}{\sin \frac{\mu\pi}{2} + \frac{n\mu}{1-\mu} \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}}$$

*with some real  $\alpha, \gamma > 0$  and  $0 < \mu \leq 1$ , then  $f(z) \in \mathcal{STS}(\mu)$ .*

*Proof.* By using Lemma 1 with  $\lambda = 1$  we deduce

$$|\arg(f'(z))| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}).$$

with

$$\alpha = \beta + \frac{2}{\pi} \arctan(n\beta).$$

Using again Lemma 1, we get

$$\left| \arg \left( \frac{f(z)}{z} \right) \right| < \frac{\pi}{2}\gamma \quad (z \in \mathbb{U}),$$

where  $\gamma$  is the solution of the equation

$$\beta = \gamma + \frac{2}{\pi} \arctan(n\gamma).$$

If we set  $p(z) = \frac{zf'(z)}{f(z)}$  and  $P(z) = \frac{f(z)}{z}$ , then we have  $p(z) \in \mathcal{H}[1, n]$  and

$$f'(z) + zf''(z) = P(z)((p(z))^2 + zp'(z)) \quad (z \in \mathbb{U}),$$

where

$$|\arg(P(z))| < \frac{\pi}{2}\gamma \quad (z \in \mathbb{U}).$$

It follows that

$$|\arg((p(z))^2 + zp'(z))| \leq |\arg(f'(z) + zf''(z))| + |\arg(P(z))| < \frac{\pi}{2}(\alpha + \gamma).$$

For the condition of  $\phi(\mu)$ , we deduce that

$$|\arg((p(z))^2 + zp'(z))| < \phi(\mu) \quad (z \in \mathbb{U})$$

implies by means of Lemma 3.

$$|\arg(p(z))| < \frac{\pi}{2}\mu \quad (z \in \mathbb{U}).$$

□

We consider an example for Theorem 1.

**Example 1.** Let us consider the function

$$f(z) = z + \sin \frac{\pi\alpha}{2} z^{n+1} \quad (z \in \mathbb{U})$$

with  $0 < \alpha \leq 1$ . If we put

$$\mu = \frac{2}{\pi} \arcsin \frac{n(n+1) \sin \frac{\pi\alpha}{2}}{(n+1)^3 - \sin^2 \frac{\pi\alpha}{2}},$$

the function  $f(z)$  satisfies the condition of Theorem 1.

Because differentiating the function  $f(z)$ , we obtain

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \frac{(n+1)(n+1 + \sin \frac{\pi\alpha}{2} z^n)}{(n+1)^2 + \sin \frac{\pi\alpha}{2} z^n} \\ &= n+1 - \frac{n}{1 + \frac{\sin \frac{\pi\alpha}{2}}{(n+1)^2} z^n} \quad (z \in \mathbb{U}) \end{aligned}$$

and therefore,

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \arcsin \frac{n(n+1) \sin \frac{\pi\alpha}{2}}{(n+1)^3 - \sin^2 \frac{\pi\alpha}{2}} \quad (z \in \mathbb{U}).$$

If we fix one of the values for  $\alpha$ ,  $\beta$  or  $\gamma$  in Theorem 1, then we can obtain others. For example, if we put  $n = 2$ ,  $\alpha = 1$  and  $\mu = \frac{1}{2}$ , then we get  $\beta = \frac{1}{2}$ ,  $\gamma = 0.227 \dots$  and the following result due to Mocanu [3].

**Corollary 1.** *If  $f(z) \in \mathcal{A}_2$  satisfies*

$$\operatorname{Re}(f'(z) + zf''(z)) > 0 \quad (z \in \mathbb{U}),$$

*then*

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{4} \quad (z \in \mathbb{U}).$$

Moreover, putting  $n = 2$ ,  $\alpha = \frac{3}{2}$  and  $\mu = 1$ , we have Corollary 2 given by Mocanu [3].

**Corollary 2.** *If  $f(z) \in \mathcal{A}_2$  satisfies*

$$|\arg(f'(z) + zf''(z))| < \frac{3}{4}\pi \quad (z \in \mathbb{U}),$$

*then*

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

Futhermore, putting  $n = 1$ , we derive Corollary 3 and Corollary 4 which were showed by Mocanu [2].

**Corollary 3.** *If  $f(z) \in \mathcal{A}$  satisfies*

$$\operatorname{Re}(f'(z) + zf''(z)) > 0 \quad (z \in \mathbb{U}),$$

*then*

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{3} \quad (z \in \mathbb{U}).$$

**Corollary 4.** *If  $f(z) \in \mathcal{A}$  satisfies*

$$|\arg(f'(z) + zf''(z))| < \frac{2}{3}\pi \quad (z \in \mathbb{U}),$$

*then*

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

#### 4. INTEGRAL VERSION OF THE RESULTS

Let us define the function

$$F(z) = \int_0^z \frac{f(t)}{t} dt \quad (z \in \mathbb{U})$$

for  $f(z) \in \mathcal{A}_n$ . This integral operator  $F(z)$  is given by Alexander [1] and is said to be Alexander integral operator.

For this Alexander integral operator for  $f(z)$ , we derive

**Theorem 2.** *If  $f(z) \in \mathcal{A}_n$  satisfies*

$$|\arg(f'(z))| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U})$$

*for*

$$\alpha = \beta + \frac{2}{\pi} \arctan(n\beta),$$

$$\beta = \gamma + \frac{2}{\pi} \arctan(n\gamma),$$

$$\frac{\pi}{2}(\alpha + \gamma) \leq \phi(\mu)$$

*and*

$$\phi(\mu) = \frac{\pi}{2}(\mu + 1) - \arctan \frac{\cos \frac{\mu\pi}{2}}{\sin \frac{\mu\pi}{2} + \frac{n\mu}{1-\mu} \left( \frac{1-\mu}{1+\mu} \right)^{\frac{1+\mu}{2}}}$$

*with some real  $\alpha, \gamma > 0$  and  $0 < \mu \leq 1$ , then the Alexander integral operator  $F(z)$  of  $f(z)$  belongs to the class  $\mathcal{STS}(\mu)$ .*

The proof of Theorem 2 follows by replacing  $f(z)$  with  $F(z)$  in Theorem 1.

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